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Problems for *Physik der Materie III*

Due by June 5, 2019

Series 7: Free electron Fermi gas and quantum statistics

7.1 Mean kinetic energy

Determine the average energy per electron $\bar{E} = U/N$ in a three-dimensional free electron gas at $T = 0$ K, where U is the total energy and N the number of electrons. Which temperature T has to be reached in a classical ideal gas to obtain the same mean kinetic energy as in a Fermi gas with $E_F = 5$ eV ?

7.2 Fermi parameters of metals

The Fermi energy E_F of a three-dimensional free electron gas can be written as:

$$E_F = \frac{\hbar^2}{2m}(3\pi^2 n_{el})^{\frac{2}{3}},$$

where m is the electron mass and n_{el} the electron density.

(a) Derive expressions for the Fermi-wave vector k_F and the Fermi-velocity v_F in terms of n_{el} and m.

(b) Assuming that each atom contributes an integer number of electrons to the electron gas of a metal, calculate E_F (in eV), T_F (in K), k_F (in cm^{-1}), and v_F (in cm/s) for Cu and Al with atomic densities $n_{at}^{Cu} = 8.45 \cdot 10^{22}$ cm^{-3} and $n_{at}^{Al} = 6.02 \cdot 10^{22}$ cm^{-3}. Take into account that Cu and Al possess different numbers of valence electrons.

(c) Determine the dimensionless radius parameter $r_s = r_0/a_H$ for Cu and Al, where a_H is the Bohr radius and r_0 is the radius of the sphere that contains one electron.

(d) Calculate mean free paths l at room and liquid helium temperature for Cu. Relaxation times are $\tau^{Cu}(T = 300$ K$)\approx 2 \cdot 10^{-14}$ s and $\tau^{Cu}(T = 4.2$ K$)\approx 10^5 \cdot \tau(T = 300$ K$)$. What is the physical origin of the difference between these mean free paths?

7.3 Fermi-Dirac and Bose-Einstein distributions

The Fermi-Dirac and Bose-Einstein distributions describe how non-interacting indistinguishable particles occupy a set of available discrete energy states ϵ_i, at thermodynamic equilibrium at temperature T. The Fermi-Dirac distribution,

$$f_{FD}(\epsilon_i, T) = \frac{n_i}{g_i} = \frac{1}{\exp\left[(\epsilon_i - \mu)/k_B T\right] + 1}, \tag{1}$$

and Bose-Einstein distribution,

$$f_{BE}(\epsilon_i, T) = \frac{n_i}{g_i} = \frac{1}{\exp\left[(\epsilon_i - \mu)/k_B T)\right] - 1}, \tag{2}$$

describe the ratio $\frac{n_i}{g_i}$ of the number of n_i particles with a given energy ϵ_i to the number of quantum states g_i having energy ϵ_i. The chemical potential μ (see equations 1 and 2) denotes the change of the free energy of the system upon the addition or removal of a particle. The Fermi-Dirac distribution is valid for fermions like, e.g., conduction electrons in a metal. Fermions obey the Pauli exclusion principle. The Bose-Einstein distribution is valid for bosons like, e.g., phonons in a crystal. For bosons there is no restriction how many of them can occupy the same quantum state.

(a) Under which conditions do the Fermi-Dirac and Bose-Einstein distributions approach the Boltzmann distribution,

$$f_B(\epsilon_i, T) = n_i/g_i = \exp\left[-(\epsilon_i - \mu)/k_B T\right], \tag{3}$$

which is valid for classical particles?

(b) Determine n_i/g_i for the special case $\epsilon_i = \mu$.

(c) Sketch the distributions as function of $(\epsilon_i - \mu)/k_B T$ in the range of -5 to $+5$ and comment on your result.

(d) Plot and discuss n_i/g_i as a function of ϵ_i in the range 0.1 eV $\leq \epsilon_i - \mu \leq +0.1$ eV for the two temperatures $T = 10^{-3}$ and 300 K.